By Jacob Alperin-Sheriff

# Discrete Gaussian Sampling-Techniques and Dangers

04/21/2017

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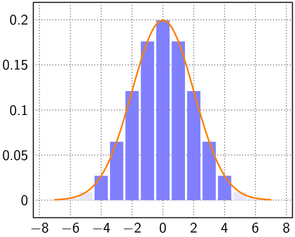
- Close enough for LWE - small number of samples
- For (SIS-based) signatures - large number of samples per instance
- Can't just approximate

# Discrete Gaussian Distribution

- Discrete Gaussian $D_{\mathbb{Z},\sigma}$ for $\sigma = 2$



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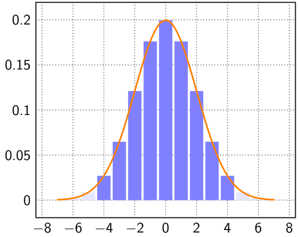

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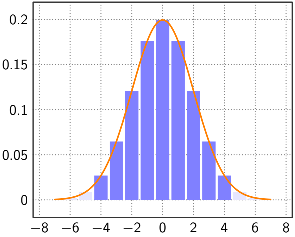


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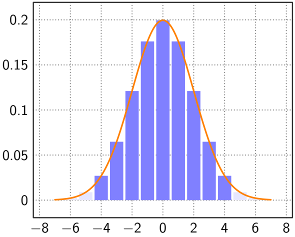


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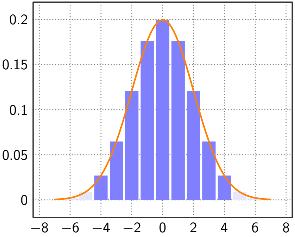


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- Actual sampling: ignore the (very unlikely) points outside $[-\tau\sigma, \tau\sigma]$

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- **Verify**$((\mathbf{z}, \mathbf{c}), \mu)$:
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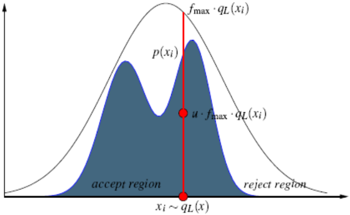
- Key Step: rejection sampling – hides $\mathbf{S}$ contribution to signature.

# Rejection Sampling

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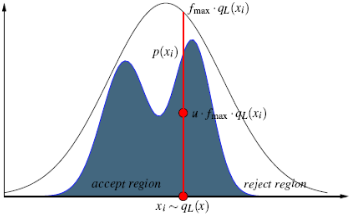


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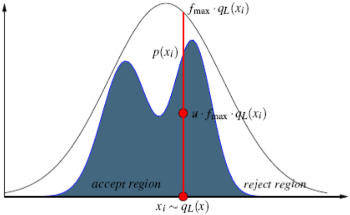


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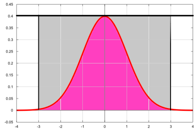
1. Sample $Y \leftarrow g$
2. Accept $Y$ with probability $\min(f(Y)/(Mg(Y), 1)$.
   - Need $f(x) \leq Mg(x)$ (except with negligible probability over $x$)

# Rejection Sampling for Discrete Gaussian Distributions

- For param $\sigma$, sample probabilities must be proportional to

$$\rho_\sigma(x) = \exp(-x^2/(2\sigma^2))$$



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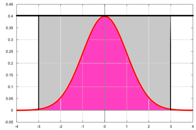

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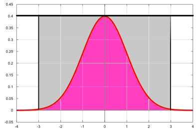

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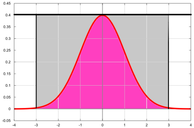

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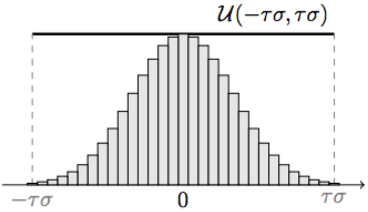
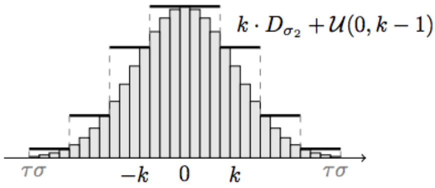
- Problems:
  - High rejection rate
  - Computing $\rho_\sigma$ to high precision is expensive

# Bernoulli Rejection Sampling [DDLL' 12]




(a) from uniform distribution (repetition rate $\approx 10$)




(b) from our adapted distribution (repetition rate $\approx 1.47$)

- "Core sampler" of $D_{\sigma_2}^{+}$ where $\sigma_2 = \sqrt{1/(2\ln(2))}$.
  - ⋆ $\rho_{\sigma_2}(x) = 2^{-x^2}, x \in \mathbb{Z}$
  - ⋆ In DDLL'12, binary-style rejection sampler given access to uniform bits.

# Bernoulli Rejection-Core Sampler

## Sampling $D^{+}_{\sigma_2}$

Draw random bit $b$.
**if** random bit $b = 0$ **then** return $0$
**for** $i = 1$ to $\infty$ **do**
  Draw random bits $b_1, \ldots, b_k$ for $k = 2i - 1$
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- Why it works: binary expansion of $\rho_{\sigma_2}(\{0, \ldots, j\})$ is

$$\rho_{\sigma_2}(0, \ldots, j) = \sum_{i=0}^{j} 2^{-i^2} = 1.100100001\underbrace{0\ldots0}_{6}1\ldots\underbrace{0\ldots0}_{2(j-1)}1$$

# Bernoulli Rejection (Full Algorithm)

**Sampling $D^+_{k\sigma_2}$, $k \in \mathbb{Z}$**

Sample $x \leftarrow D^+_{\sigma_2}$.
Sample $y \leftarrow \{0, \ldots, k-1\}$.
Let $z \leftarrow kx + y$.
Sample $b$ with probability $\exp(-y(y + 2kx)/(2(k\sigma_2)^2))$
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- Sampling the exponential distribution can be done efficiently
  - Takes time $O(\log k)$.
  - Needs small lookup table with

$$\mathsf{ET}[i] := \exp(-2^i/(2(k\sigma_2)^2)), i \in [0, O(\log k)]$$

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- Problem-Information Revealed by Timing Attacks!
- When **for** loop not entered, algorithm always outputs $0$
- Algorithm for $D^+_{\sigma_2}$ is slow in worst case.
- Can mitigate with batching

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- Can be sped up with additional guide table
- Problems: Table is quite large; infeasible for constrained devices.

# Knuth-Yao Sampling

START

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|---|---|
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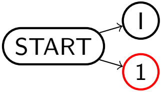


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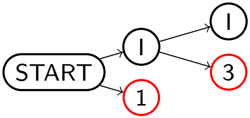


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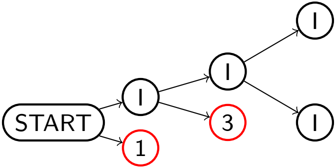


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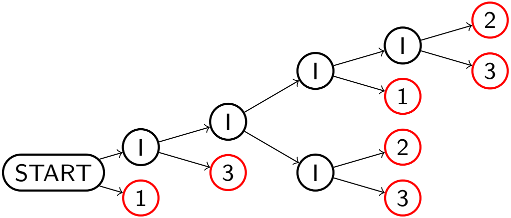


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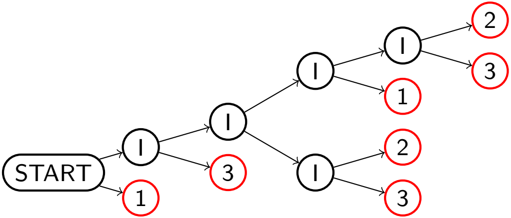


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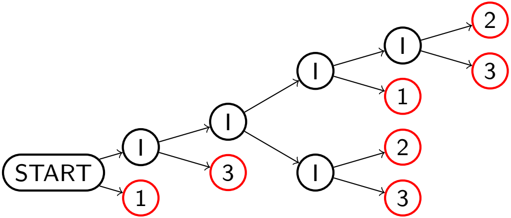


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- Designed to minimize (average) number of bits required to sample
- Theorem: Knuth-Yao requires at most 2 more than entropy of dist.

# Knuth-Yao Sampling (Gaussians) [Dwarakanath/Galbraith]

| x | $\sigma_{10}(x)$ | Binary expansion of $\sigma_{10}(x)$ |
|---|---|---|
| 0 | 0.01994711 | 0.00000101000110110100 |
| $\pm 1$ | 0.03969525 | 0.00001010001010010111 |
| $\pm 10$ | 0.02419707 | 0.00000110001100011100 |
| $\pm 20$ | 0.00539909 | 0.00000001011000011101 |
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| $\pm 40$ | 0.00001338 | 0.00000000000000001110 |

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  2. Pick a set
  3. Perform Knuth-Yao within the set

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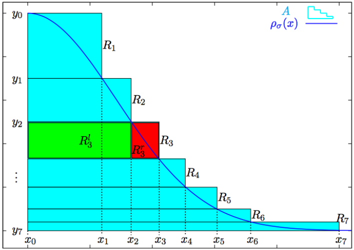
| $\pm 20$ | 0.00539909 | 0.00000001011000011101 |
| $\pm 30$ | 0.00044318 | 0.00000000000111010001 |
| $\pm 40$ | 0.00001338 | 0.00000000000000001110 |

- Need to store table of probabilities, which is large.
- Can cut down by performing Knuth-Yao in "blocks"
  1. Partition into disjoint sets with almost equivalent probabilities
  2. Pick a set
  3. Perform Knuth-Yao within the set
- Knuth-Yao is not constant time!
- Can be mitigated by batching

# Discrete Ziggurat Sampling

1. Partition density function into $m$ rectangles of equal probability



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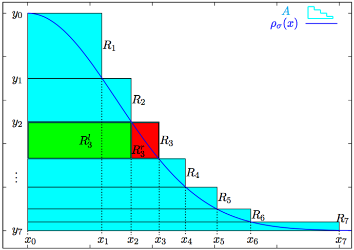

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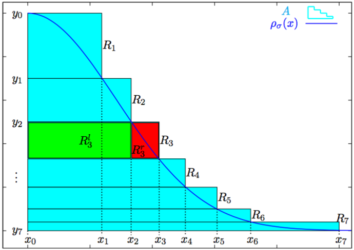

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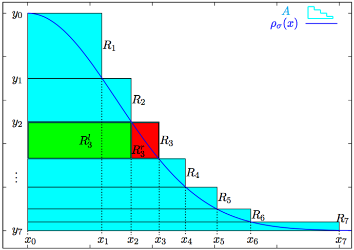

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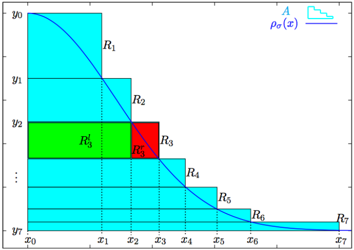


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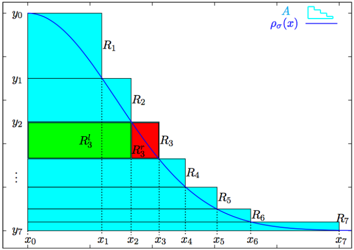


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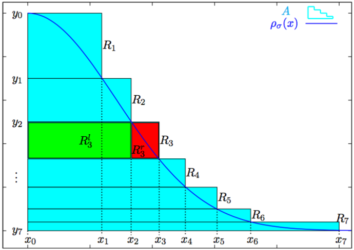


- Sampling in discrete case requires some care
    - ⋆ Partitioning can't be done by "area", but by probability
- No clear vulnerability to timing attacks.

# Batching

- Technique to make algorithm (e.g. Knuth-Yao) constant time

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- Have algorithm run in "time" proportional to $cn + n$ being used.
- Pitfall: Making sure implementation is constant time is extremely hard

# Walter-Micciancio Sampling

$\textsc{SampleZ}_{b,k,\max}(c, \sigma)$
  $x \leftarrow \textsc{SampleI}(\max)$
  $K \leftarrow \sqrt{\sigma^2 - \bar{\sigma}^2}/\sigma_{\max}$
  $c' \leftarrow \lfloor c + Kx \rceil_k$
  $y \leftarrow \textsc{SampleC}_{b,\sigma_0}(c')$
  **return** $y$

$\textsc{SampleI}(i)$
  **if** $i = 0$
    $x \leftarrow \textsc{SampleB}_{\sigma_0}(0)$
    **return** $x$
  $x_1 \leftarrow \textsc{SampleI}(i-1)$
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Algorithm 1: A sampling algorithm for $\mathcal{D}_{\mathbb{Z}+c,\sigma}$ for arbitrary $c$ and $\sigma$. Definitions for $z_i$ and $\sigma_i$ as in (3) and (4) and $\bar{\sigma}$ as in (5). SampleB is an arbitrary base sampler for fixed $\sigma_0$ and small number of cosets $c + \mathbb{Z}$, where $c \in \mathbb{Z}/b$.

$$z_i = \lfloor \sigma_{i-1}/\sqrt{2\eta_\epsilon(\mathbb{Z})} \rfloor, \sigma_i^2 = (z_i^2 + \max((z_i - 1)^2, 1))\sigma_{i-1}^2$$

- New algorithm with constant-time online phase

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```
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  K ← √(σ² − σ̄²)/σ_max
  c′ ← ⌊c + Kx⌉_k
  y ← SampleC_{b,σ0}(c′)
  return y

SampleI(i)
  if i = 0
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  x1 ← SampleI(i − 1)
  x2 ← SampleI(i − 1)
  y = z_i x1 + max(1, z_i − 1) x2
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SampleC_b(c ∈ b^{−k}ℤ)
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  return g + SampleC_b(c − g ∈ b^{−k+1}ℤ)
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- Works by recursively combining samples with smaller $\sigma$s.
- Assumes access to $D_{\mathbb{Z}+c,\sigma_0}$ with small $\sigma_0 \geq \sqrt{2}\eta_\epsilon(\mathbb{Z})$
  - Authors suggest generating these offline in “idle times”
  - Doesn't seem plausible for constrained devices
  - Relies on idle time (frequent queries could eliminate it)

# Walter-Micciancio (Runtime)

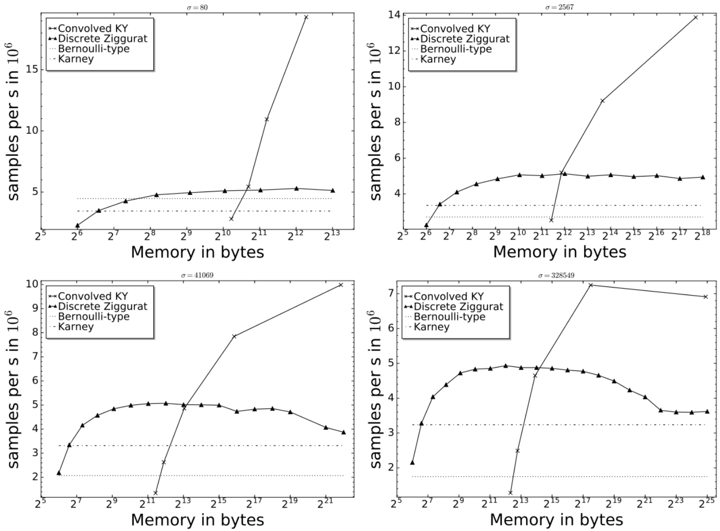


Figure 1: Time memory trade-off for combination sampler ("Convolved KY") and discrete Ziggurat compared to Bernoulli-type sampling for $\sigma \in \{2^5, 2^{10}, 2^{14}, 2^{17}\}\sqrt{2\pi}$. Knuth-Yao corresponds to right most point of "Convolved KY".

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- FLUSH+RELOAD must be run on same system as crypto

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- Would be nice to see concrete implementations without them